# THE MOMENT OF THE GRAVITATIONAL FORCES ACTING ON A SATELLITE

(MOMENT GRAVITATSIONNYKH SIL, DEISTVUIUSHCHIKH NA SPUTNIK)

*PMM Vol.27, No.2, 1963, pp. 377-378*

A. I. LUR'E
(Leningrad)

*(Received November 29, 1962)*

The moment acting on a satellite due to the forces of the geodetic gravitational field were calculated in [1]. In this calculation, the author proceeded from an expression for the potential energy of a small body embedded in this force field that was obtained in [2]. In the present note we propose a direct derivation that leads to comparatively compact formulas.

1. We begin by examining a central force field - the gravitational field of a spherical earth. Let $\mathbf{r} = OC$ be the radius vector for the mass center $C$ of the satellite, with the origin $O$ at the center of the earth; let the vector $\rho = CM$ determine the position of a mass element $dm$ in the satellite; and let the gravity force acting on this mass be equal to $\mathbf{F}dm$ (limited to terms linear in $\rho$)

$$\mathbf{F}dm = -\frac{\mu dm}{|\mathbf{r}+\rho|^3}(\mathbf{r}+\rho) = -\frac{\mu dm}{r^3}\mathbf{r} + \frac{\mu}{r^3}\left(3\frac{\mathbf{r}\cdot\rho}{r^2}\mathbf{r} - \rho\right) \tag{1.1}$$

Here $\mu$ is the product of the mass of the earth and the gravitational constant. Hence the resultant moment of the forces $\mathbf{F}dm$ with respect to the mass center may be represented in the form*

$$\mathbf{m}^{(c)} = \int \varrho \times \mathbf{F}dm = \frac{3\mu}{r^5}\mathbf{r}\cdot\int \varrho\varrho\, dm \times \mathbf{r} \tag{1.2}$$

---

* In the sequel $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{ab}$ denote the scalar, vector, and dyadic products respectively of the vectors $\mathbf{a}$ and $\mathbf{b}$.

where the integration is carried out over the entire mass. Here, we have taken into account the fact that $C$ is the center of mass, so that

$$\int \rho dm = 0 \tag{1.3}$$

We recall now the definition of the inertia tensor of a body at a point $C$

$$\Theta^c = \int (E\varrho \cdot \rho - \varrho\varrho)\, dm = E\vartheta - \int \varrho\varrho\, dm \tag{1.4}$$

where $E$ is the unit tensor, and $\vartheta$ is one-half the first invariant (sum of the diagonal elements) of the tensor $\Theta^c$. Hence, we readily obtain

$$\mathbf{m}^c = -\frac{3\mu}{r^5}\mathbf{r}\cdot\Theta^c\times\mathbf{r} = -\frac{3\mu}{r^3}\mathbf{e}_r\cdot\Theta^c\times\mathbf{e}_r \tag{1.5}$$

where $\mathbf{e}_r$ is a unit vector in the direction of $\mathbf{r}$ (external vertical). Now, denoting the unit vectors along the principal axes of the tensor $\Theta^c$ by $\mathbf{i}_s$, we find the following expressions for the moments with respect to these axes

$$\mathbf{m}^c\cdot\mathbf{i}_s = -\frac{3\mu}{r^3}\mathbf{e}_r\cdot\Theta^c\cdot(\mathbf{e}_r\times\mathbf{i}_s) \tag{1.6}$$

Entering in the resulting formulas are the cosines of the angles made by the axes $\mathbf{i}_s$ with the vertical.

2. The same method of calculation may be repeated when a nonspherical earth is taken into account. If only the second harmonic is retained in the expansion of the potential energy of the gravity forces, then, as is known [3], the expression (1.1) for the gravity forces has to be supplemented by the terms

$$-dm\frac{\mu R_0^2\varepsilon}{r'^5}\left[\mathbf{r}' - \frac{5}{r'^2}(\mathbf{k}\cdot\mathbf{r}')^2\mathbf{r}' + 2\mathbf{k}\cdot\mathbf{r}'\mathbf{r}'\right],\ \mathbf{r}' = \mathbf{r} + \varrho \tag{2.1}$$

Here $\mathbf{k}$ is a unit vector perpendicular to the plane of the equator, $R_0$ is the equatorial radius, and $\varepsilon$ is a constant figure of the earth ($\varepsilon \approx 0.00164$). Limiting the expansion (2.1) to terms linear in $\rho$, the expression (2.1) is brought to the form

$$dm\frac{5\mu R_0^2\varepsilon}{r^5}\left[\mathbf{e}_r\cdot\rho\mathbf{e}_r(1-7\cos^2\theta) + 2\cos\theta(\mathbf{k}\cdot\rho\mathbf{e}_r + \mathbf{e}_r\cdot\rho\mathbf{k}) - \frac{2}{5}\mathbf{k}\cdot\rho\mathbf{k}\right] + * \tag{2.2}$$

in which the asterisk denotes terms that are parallel to $\rho$ but independent of $\rho$ (these drop out in the calculation of the moment); and $\mathbf{k}\times\mathbf{e}_r = \cos\theta$, where $\theta$ is the latitude reckoned from the pole of the equatorial plane. From formulas (1.2) and (1.4) it is now easy to obtain

$$\mathbf{m}^c = -\frac{3\mu}{r^3}\mathbf{e}_r\cdot\boldsymbol{\Theta}^c\times\mathbf{e}_r - \frac{5\mu R_0^2\varepsilon}{r^5}\,[(1-7\cos^2\theta)\,\mathbf{e}_r\cdot\boldsymbol{\Theta}^c\times\mathbf{e}_r +$$
$$+ 2\cos\theta\,(\mathbf{k}\cdot\boldsymbol{\Theta}^c\times\mathbf{e}_r + \mathbf{e}_r\cdot\boldsymbol{\Theta}^c\times\mathbf{k}) + 4\dot\theta\mathbf{e}_r\times\mathbf{k}\cos\theta - \frac{2}{5}\,\mathbf{k}\cdot\boldsymbol{\Theta}^c\times\mathbf{k}] \qquad (2.3)$$

In order to calculate the moment of the forces with respect to the principal axes of the satellite, it is likewise necessary to know the orientation of these axes with respect to the vector **k**

$$\mathbf{m}^c\cdot\mathbf{i}_s = -\frac{3\mu}{r^3}\mathbf{e}_r\cdot\boldsymbol{\Theta}^c\cdot(\mathbf{e}_r\times\mathbf{i}_s) - \frac{5\mu R_0^2\varepsilon}{r^5}\,\{(1-7\cos^2\theta)\,\mathbf{e}_r\cdot\boldsymbol{\Theta}^c\cdot(\mathbf{e}_r\times\mathbf{i}_s) +$$
$$+ 2\cos\theta\,[\mathbf{k}\cdot\boldsymbol{\Theta}^c\cdot(\mathbf{e}_r\times\mathbf{i}_s) + \mathbf{e}_r\cdot\boldsymbol{\Theta}^c\cdot(\mathbf{k}\times\mathbf{i}_s)] - 2\dot\theta\sin 2\theta\,\mathbf{n}\cdot\mathbf{i}_s - \frac{2}{5}\,\mathbf{k}\cdot\boldsymbol{\Theta}^c\cdot(\mathbf{k}\times\mathbf{i}_s)\} \qquad (2.4)$$

Here **n** is a unit vector along the direction $\mathbf{e}_r \times \mathbf{k}$.

## BIBLIOGRAPHY


1. Roberson, R.E., Gravitational torque on a satellite vehicle. *Journ. Frank. Inst.* 265, 13-22, 1958.

2. Roberson, R.E. and Tatistcheff, D., The potential energy of a small rigid body in the gravitational field of an oblate spheroid. *Journ. Frank. Inst.* 262, 209-214, 1956.

3. Lur'e, A.I., *Analiticheskaia mekhanika (Analytical Mechanics)*. Fizmatgiz, 1961.


*Translated by* E.E.Z.